Distribution-Dependent Stochastic Functional Differential Equations *

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Abstract

By the approximation method introduced in [14], the existence and uniqueness are proved for a class of distribution-dependent stochastic functional differential equations (DDSFDEs). Moreover, combining the Harnack and shift-Harnack inequalities for classical stochastic functional differential equations with Girsanov's theorem, Harnack and shift-Harnack inequalities are obtained for the non-linear semigroup P_t^* associated to the functional solution.

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1 Introduction

As we know, the distribution for the degenerate SDEs satisfies a Fokker-Planck equation in the distribution sense (see [17, 1.3] for details). On the other hand, some special functional SDEs can be treated as the degenerate SDEs in a higher dimensional space. More precisely, considering the following SFDEs:

$$(1.1)\quad \mathrm{d}X(t) = b\left(t, X(t), \int_{-\tau}^{t} F(s, X(s))\mathrm{d}s\right)\mathrm{d}t + \sigma\left(t, X(t), \int_{-\tau}^{t} F(s, X(s))\mathrm{d}s\right)\mathrm{d}W(t),$$

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let $Y(t)=\int_{-\tau}^{t}F(s,X(s))\mathrm{d}s$, then we have

$$
\begin{aligned}
&\mathrm{d}Y(t)=F(t,X(t))\mathrm{d}t,\\
&\mathrm{d}X(t)=b\left(t,X(t),Y(t)\right)\,\mathrm{d}t+\sigma\left(t,X(t),Y(t)\right)\,\mathrm{d}W(t).
\end{aligned}
\tag{1.2}
$$

This implies that the distribution of $(\int_{-\tau}^{t}F(s,X(s))\mathrm{d}s,X(t))$ satisfies Fokker-Planck equation. If the coefficients depend on the distribution, i.e.

$$
\begin{aligned}
\mathrm{d}X(t)=&\,b\left(t,X(t),\int_{-\tau}^{t}F(s,X(s),\mathscr{L}_{X(s)})\mathrm{d}s,\mathscr{L}_{(\int_{-\tau}^{t}F(s,X(s),\mathscr{L}_{X(s)}),X(t))\mathrm{d}s}\right)\,\mathrm{d}t\\
&+\sigma\left(t,X(t),\int_{-\tau}^{t}F(s,X(s),\mathscr{L}_{X(s)})\mathrm{d}s,\mathscr{L}_{(\int_{-\tau}^{t}F(s,X(s),\mathscr{L}_{X(s)})\mathrm{d}s,X(t))}\right)\,\mathrm{d}W(t).
\end{aligned}
\tag{1.3}
$$

letting $Y(t)=\int_{-\tau}^{t}F(s,X(s),\mathscr{L}_{X(s)})\mathrm{d}s$, we obtain

$$
\begin{aligned}
&\mathrm{d}Y(t)=F(t,X(t),\mathscr{L}_{X(t)})\mathrm{d}t,\\
&\mathrm{d}X(t)=b\left(t,X(t),Y(t),\mathscr{L}_{(Y(t),X(t))}\right)\,\mathrm{d}t+\sigma\left(t,X(t),Y(t),\mathscr{L}_{(Y(t),X(t))}\right)\,\mathrm{d}W(t).
\end{aligned}
\tag{1.4}
$$

By Itô formula and integration by parts formula, it is easy to see that the distribution of $(\int_{-\tau}^{t}F(s,X(s),\mathscr{L}_{X(s)})\mathrm{d}s,X(t))$ is a solution to some non-linear PDE. (see [14] for details)

In this paper, we investigate general DDSFDEs (1.5),

and we aim to extend the results of [14] to the present case. More precisely, we prove the existence and uniqueness of solutions by the same approximation introduced in [14], and obtain the Harnack and shift-Harnack inequalities for the non-linear semigroup P_t^* associated to the functional solution.

The main difference for the delay case is that $\mathbb{E}|X(t)|^p$ can not be obtained directly as in the case without delay. Instead, we always estimate $\mathbb{E}\sup_{s\in[-r_0,t]}|X(s)|^p$, see the proof of Theorem 2.1. Moreover, to ensure the coupling succeed in coupling by change of measure, we should keep the delay part in the coupling equations, which leads to more difficult estimate for $\mathbb{E}R\log R$ and $\mathbb{E}R^{\frac{p}{p-1}}$, see the proof of [7, Theorem 4.31, 4.32]. Meanwhile, we assume the Lipschitzian condition on the drift with delay instead of the weaker growth condition for the drift without delay. Furthermore, due to the delay, the condition for the existence and uniqueness of stationary distribution is non-trivial, in other words, we can not obtain the similar assertions in [14, Theorem 3.1] under the same conditions. Thus, we leave the study of exponential ergodicity in the future.

Fix a constant $r_0>0$, let $\mathscr{C}=C\left([-r_0,0];\mathbb{R}^d\right)$ be equipped with the uniform norm defined as $\|\xi\|_\infty:=\sup_{s\in[-r_0,0]}|\xi(s)|$, $\xi\in\mathscr{C}$. Then $(\mathscr{C},\|\cdot\|_\infty)$ is a Polish space. For any $f\in C\left([-r_0,\infty);\mathbb{R}^d\right)$ and $t\geq 0$, let $f_t(\theta)=f(t+\theta)$, $\theta\in[-r_0,0]$, then $f_t\in\mathscr{C}$.

Consider the following DDSFDE with delay on $\mathbb{R}^d$:

$$
\mathrm{d}X(t)=b(t,X_t,\mathscr{L}_{X_t})\,\mathrm{d}t+\sigma(t,X_t,\mathscr{L}_{X_t})\,\mathrm{d}W(t),
\tag{1.5}
$$

where $W=(W(t))_{t\geq 0}$ is an d-dimensional standard Brownian motion with respect to a complete filtration probability space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$, $\mathscr{L}_{X_t}$ is the distribution of X_t. Let $\mathscr{P}(\mathscr{C})$ denote the space of all probability measures on $\mathscr{C}$,

$$
b:[0,\infty)\times\mathscr{C}\times\mathscr{P}(\mathscr{C})\to\mathbb{R}^d;\quad \sigma:[0,\infty)\times\mathscr{C}\times\mathscr{P}(\mathscr{C})\to\mathbb{R}^d\otimes\mathbb{R}^d.
$$

In this paper, we investigate the existence and uniqueness of strong to (1.5) and characterize their distribution properties.

When more than one probability measures on Ω are concerned, we use $\mathscr{L}_{X_t}|_{\mathbb{P}}$ instead of $\mathscr{L}_{X_t}$ to emphasize the distribution under probability $\mathbb{P}$. Due to technical reasons, we will restrict ourselves to the following subspace of $\mathscr{P}(\mathscr{C})$:

$$\mathscr{P}_2 := \left\{ \nu \in \mathscr{P}(\mathscr{C}) : \ \nu(\|\cdot\|_\infty^2) := \int_{\mathscr{C}} \|\xi\|_\infty^2 \nu(\mathrm{d}\xi) < \infty \right\},$$

which is a Polish space under the Wasserstein distance

$$\mathbb{W}_2(\mu_1, \mu_2) := \inf_{\pi \in \mathscr{C}(\mu_1,\mu_2)} \left(\int_{\mathscr{C}\times\mathscr{C}} \|\xi - \eta\|_\infty^2 \pi(\mathrm{d}\xi, \mathrm{d}\eta) \right)^{\frac{1}{2}}, \quad \mu_1, \mu_2 \in \mathscr{P}_2,$$

where $\mathscr{C}(\mu_1, \mu_2)$ is the set of all couplings for μ_1 and μ_2.

Definition 1.1. (1) For any $s \geq 0$, a continuous adapted $\mathscr{C}$-valued process $(X_t^s)_{t\geq s}$ is called a (strong) solution of (1.5) from time s if

$$\int_s^t \mathbb{E}\big\{|b(u, X_u^s, \mathscr{L}_{X_u^s})| + \|\sigma(u, X_u^s, \mathscr{L}_{X_u^s})\|^2\big\}\mathrm{d}u < \infty, \quad t > s,$$

and $\mathbb{P}$-a.s.,

$$X^s(t) = X^s(t \wedge s) + \int_s^{t\vee s} b(u, X_u^s, \mathscr{L}_{X_u^s})\mathrm{d}u + \int_s^{t\vee s} \sigma(u, X_u^s, \mathscr{L}_{X_u^s})\mathrm{d}W(u), \quad t \geq s - r_0.$$

We say that (1.5) has (strong or pathwise) existence and uniqueness, if for any $s \geq 0$ and $\mathscr{F}_s$-measurable random variable X_s^s with $\mathbb{E}\|X_s^s\|_\infty^2 < \infty$, the equation from time s has a unique solution $(X_t^s)_{t\geq s}$. For simplicity, we denote $X_t^0 = X_t$.

(2) A couple $(\tilde{X}_t, \tilde{W}(t))_{t\geq s}$ is called a weak solution to (1.5) from time s, if $\tilde{W}_t$ is a d-dimensional standard Brownian motion with respect to a complete filtration probability space $(\tilde{\Omega}, \{\tilde{\mathscr{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$, and $\tilde{X}_t$ solves the DDSDE with delay

$$\mathrm{d}\tilde{X}(t) = b(t, \tilde{X}_t, \mathscr{L}_{\tilde{X}_t}|_{\tilde{\mathbb{P}}})\mathrm{d}t + \sigma(t, \tilde{X}_t, \mathscr{L}_{\tilde{X}_t}|_{\tilde{\mathbb{P}}})\mathrm{d}\tilde{W}(t), \quad t \geq s. \tag{1.6}$$

(3) (1.5) is said to have weak uniqueness if for any $s \geq 0$, any two weak solutions of the equation from time s with common initial distribution in $\mathscr{P}_2$ are equal in law. Precisely, if $s \geq 0$ and $(\bar{X}_t^s, \bar{W}(t))_{t\geq s}$ with respect to $(\bar{\Omega}, \{\bar{\mathscr{F}}_t\}_{t\geq 0}, \bar{\mathbb{P}})$ and $(\tilde{X}_t^s, \tilde{W}(t))_{t\geq s}$ with respect to $(\tilde{\Omega}, \{\tilde{\mathscr{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$ are weak solutions of (1.5), then $\mathscr{L}_{\bar{X}_s^s}|_{\bar{\mathbb{P}}} = \mathscr{L}_{\tilde{X}_s^s}|_{\tilde{\mathbb{P}}}$ implies $\mathscr{L}_{\bar{X}_\cdot^s}|_{\bar{\mathbb{P}}} = \mathscr{L}_{\tilde{X}_\cdot^s}|_{\tilde{\mathbb{P}}}$.

When (1.5) has strong existence and uniqueness, the solution $(X_t)_{t\geq 0}$ is a Markov process in the sense that for any $s \geq 0$, $(X_t)_{t\geq s}$ is determined by solving the equation from time s with initial state X_s. More precisely, letting $\{X_t^{s,\xi}\}_{t\geq s}$ denote the solution of the equation from time s with initial state $X_s^s = \xi$, the existence and uniqueness imply

$$X_t^{s,\xi} = X_t^{u, X_u^{s,\xi}}, \quad t \geq u \geq s \geq 0, \xi \text{ is } \mathscr{F}_s\text{-measurable with } \mathbb{E}\|\xi\|_\infty^2 < \infty. \tag{1.7}$$

When the DDSDDE also has weak uniqueness, we may define a semigroup $(P_{s,t}^*)_{t\geq s}$ on $\mathscr{P}_2$ by letting $P_{s,t}^*\mu = \mathscr{L}_{X_t^s}$ for $\mathscr{L}_{X_s^s} = \mu \in \mathscr{P}_2$. Indeed, by (1.7) we have

$$P_{s,t}^* = P_{u,t}^* P_{s,u}^*, \quad t \geq u \geq s \geq 0. \tag{1.8}$$

We simply denote $P_t^* = P_{0,t}^*$.

As explained above, due to the distribution-dependence of coefficients, the semigroup $P_{s,t}^*$ is non-linear, however, we may also investigate the properties of it.

The reminder of the paper is as follows: In Section 2, we investigate the existence, uniqueness and time-space continuity of solutions. In Sections 3 and 4, we use Girsanov's theorem and the result of the classical SFDEs to establish Harnack and shift Harnack inequalities.

2 Existence and Uniqueness

In this section, we will iterate (1.5) in distributions by the same method in [14, Lemma 2.1]. we need the following assumptions:

$(H1)$ (Continuity) For every $t \geq 0$, $b(t, \cdot, \cdot)$ is continuous on $\mathscr{C} \times \mathscr{P}_2$. Moreover, there exists an increasing function $K_0 \in C([0,\infty);(0,\infty))$ such that

$$\|\sigma(t,\xi,\mu) - \sigma(t,\eta,\nu)\|_{HS}^2 \leq K_0(t)\big\{\|\xi-\eta\|_\infty^2 + \mathbb{W}_2(\mu,\nu)^2\big\}, \quad t \geq 0; \xi,\eta \in \mathscr{C}; \mu,\nu \in \mathscr{P}_2.$$

$(H2)$ (Monotonicity) There exists an increasing function $K_1 \in C([0,\infty);(0,\infty))$ such that

$$\begin{aligned}&2\langle b(t,\xi,\mu) - b(t,\eta,\nu), \xi(0)-\eta(0)\rangle + \|\sigma(t,\xi,\mu) - \sigma(t,\eta,\nu)\|_{HS}^2\\ &\leq K_1(t)\big\{\|\xi-\eta\|_\infty^2 + \mathbb{W}_2(\mu,\nu)^2\big\}, \quad t \geq 0; \xi,\eta \in \mathscr{C}; \mu,\nu \in \mathscr{P}_2.\end{aligned}$$

$(H3)$ (Growth) b is bounded on bounded sets in $[0,\infty) \times \mathscr{C} \times \mathscr{P}_2$, and there exists an increasing function $K_2 \in C([0,\infty);(0,\infty))$ such that

$$|b(t,0,\mu)|^2 + \|\sigma(t,0,\mu)\|^2 \leq K_2(t)\big\{1 + \mu(\|\cdot\|_\infty^2)\big\}, \quad t \geq 0, \mu \in \mathscr{P}_2.$$

2.1 Main results

Theorem 2.1. *Assume* $(H1)$-$(H3)$.

(1) *For any $s \geq 0$, (1.5) has strong solution X_t^s. Moreover, for any $p \geq 1$, there exists an increasing function $H_p : [0,\infty) \to (0,\infty)$ such that*

$$\mathbb{E} \sup_{t\in[s,T]} \|X_t^s\|_\infty^{2p} \leq H_p(T)(1 + \mathbb{E}\|X_s^s\|_\infty^{2p}), \quad T \geq t \geq s \geq 0. \tag{2.1}$$

(2) *For any two solutions X_t^s and Y_t^s of (1.5) with $\mathscr{L}_{X_s^s}, \mathscr{L}_{Y_s^s} \in \mathscr{P}_2$,*

$$\mathbb{E}\|X_t^s - Y_t^s\|_\infty^2 \leq 4\mathbb{E}\|X_s^s - Y_s^s\|_\infty^2 \mathrm{e}^{\int_s^t \tilde{K}(r)\mathrm{d}r}, \quad t \geq s \geq 0. \tag{2.2}$$

for some increasing function $\tilde{K} : [0,\infty) \to (0,\infty)$.

(3) (1.5) *has weak uniqueness and*

$$\mathbb{W}_2(P_t^*\mu_0, P_t^*\nu_0)^2 \le \mathbb{W}_2(\mu_0,\nu_0)^2 \mathrm{e}^{2\int_0^t K_1(s)\mathrm{d}s}, \quad t\ge 0, \mu_0,\nu_0\in\mathscr{P}_2. \tag{2.3}$$

To prove these results, we first approximate (1.5) using classical SDEs by iterating in distributions.

2.2 Proofs of Theorem 2.1

We fixed $s\ge 0$ and $\mathscr{F}_s$-measurable $\mathscr{C}$-valued random variable X_s^s with $\mathbb{E}\|X_s^s\|_\infty^2<\infty$. Let

$$X^{(0),s}(t) = X^s(s), \quad t>s; \quad X_s^{(0),s} = X_s^s, \quad \mu_t^{(0),s} = \mathscr{L}_{X_t^{(0),s}}, \quad t\ge s.$$

For any $n\ge 1$, let $(X_t^{(n),s})_{t\ge s}$ solve the classical SDE

$$\mathrm{d}X^{(n),s}(t) = b(t, X_t^{(n),s}, \mu_t^{(n-1),s})\mathrm{d}t + \sigma(t, X_t^{(n),s}, \mu_t^{(n-1),s})\,\mathrm{d}W(t), \quad X_s^{(n),s} = X_s^s, \tag{2.4}$$

where $\mu_t^{(n-1),s} = \mathscr{L}_{X_t^{(n-1),s}}$.

Lemma 2.2. *Assume (H1)-(H3). For every $n\ge 1$, the SDE (2.4) has a unique strong solution $X_t^{(n),s}$ with*

$$\mathbb{E}\sup_{t\in[s-r_0,T]}|X^{(n),s}(t)|^2<\infty, \quad T>s, n\ge 1. \tag{2.5}$$

Moreover, for any $T>0$ there exists $t_0>0$ which is independent on $s\in[0,T]$ and X_s^s, such that

$$\mathbb{E}\sup_{t\in[s,s+t_0]}|X^{(n+1),s}(t)-X^{(n),s}(t)|^2 \le 4\mathrm{e}^{-n}\mathbb{E}\sup_{t\in[s,s+t_0]}|X^{(1),s}(t)|^2, \quad s\in[0,T], n\ge 1. \tag{2.6}$$

Proof. Without loss of generality, we only prove for $s=0$.

(1) We first prove that the SDE (2.4) has a unique strong solution and (2.5) holds by induction. Assume $n=1$. Applying [7, Theorem 4.1.1] with $D=\mathbb{R}^d$, $K_n=\{x\in\mathbb{R}^d : |x|\le n\}$ and $u_n=1$, $(H1)$ and $(H2)$ yield the SDE (2.4) has a unique strong solution up to life time. It remains to prove (2.5). By $(H3)$ and Itô's formula we have

$$\begin{aligned}\mathrm{d}|X^{(1)}(t)|^2 &= 2\langle\sigma(t, X_t^{(1)}, \mu_t^{(0)})\mathrm{d}W(t), X^{(1)}(t)\rangle \\ &\quad + \big\{2\big\langle b(t, X_t^{(1)}, \mu_t^{(0)}), X^{(1)}(t)\big\rangle + \|\sigma(t, X_t^{(1)}, \mu_t^{(0)})\|_{HS}^2\big\}\mathrm{d}t.\end{aligned}$$

On the other hand, by $(H1)$-$(H3)$, there exists increasing $H\in C([0,\infty);(0,\infty))$ such that

$$\begin{aligned}&2\big\langle b(t,\xi,\mu_t^{(0)}), \xi(0)\big\rangle + \|\sigma(t,\xi,\mu_t^{(0)})\|_{HS}^2 \\ &\le 2\big\langle b(t,\xi,\mu_t^{(0)}) - b(t,0,\mu_t^{(0)}), \xi(0)\big\rangle + 2|b(t,0,\mu_t^{(0)})|\cdot|\xi(0)|\end{aligned}$$

$$
\begin{aligned}
&+2\|\sigma(t,\xi,\mu_t^{(0)})-\sigma(t,0,\mu_t^{(0)})\|_{HS}^2+2\|\sigma(t,0,\mu_t^{(0)})\|_{HS}^2\\
&\le H(t)\big\{1+\|\xi\|_\infty^2+\mu_t^{(0)}(\|\cdot\|_\infty^2)\big\},\quad t\ge 0,\xi\in\mathscr{C}.
\end{aligned}
$$

Combining these with $(H3)$ and applying the BDG inequality, for any $N\in[1,\infty)$ and $\tau_N:=\inf\{t\ge 0:|X^{(1)}(t)|\ge N\}$, we have

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[-r_0,t\wedge\tau_N]}|X^{(1)}(s)|^2\le 4\mathbb{E}\|X_0^{(1)}\|_\infty^2+2H(t)\mathbb{E}\int_0^{t\wedge\tau_N}\big(1+\|X_s^{(1)}\|_\infty^2+\mu_s^{(0)}(\|\cdot\|_\infty^2)\big)\mathrm{d}s\\
&\qquad+4H(t)\mathbb{E}\bigg(\int_0^{t\wedge\tau_N}|X^{(1)}(s)|^2\big(1+\|X_s^{(1)}\|_\infty^2+\mu_s^{(0)}(\|\cdot\|_\infty^2)\big)\mathrm{d}s\bigg)^{\frac12}\\
&\le 4\mathbb{E}\|X_0^{(1)}\|_\infty^2+\frac12\mathbb{E}\sup_{s\in[-r_0,t\wedge\tau_N]}|X^{(1)}(s)|^2\\
&+\{2H(t)+8H(t)^2\}\mathbb{E}\int_0^{t\wedge\tau_N}\big(1+\|X_s^{(1)}\|_\infty^2+\mu_s^{(0)}(\|\cdot\|_\infty^2)\big)\mathrm{d}s.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[-r_0,t\wedge\tau_N]}|X_s^{(1)}|^2\le 8\mathbb{E}\|X_0^{(1)}\|_\infty^2\\
&\qquad\qquad+\{4H(t)+16H(t)^2\}\int_0^t\Big\{1+\mathbb{E}\sup_{r\in[-r_0,s\wedge\tau_N]}|X^{(1)}(r)|^2+\mu_s^{(0)}(\|\cdot\|_\infty^2)\Big\}\mathrm{d}s.
\end{aligned}
$$

By Gronwall's lemma and letting $N\to\infty$, we arrive at

$$
\mathbb{E}\sup_{s\in[-r_0,t]}|X^{(1)}(s)|^2<\infty.
$$

Therefore, (2.5) holds for $n=1$.

Now, assume that the assertion holds for $n=k$ for some $k\ge 1$, we intend to prove it for $n=k+1$. This can be done in the same way by using $(X_\cdot^{(k+1)},\mu_\cdot^{(k)},X_\cdot^{(k)})$ in place of $(X_\cdot^{(1)},\mu_\cdot^{(0)},X_\cdot^{(0)})$. So, we omit the proof to save space.

(2) To prove (2.6), for $n\ge 1$ we simply denote

$$
\begin{aligned}
&\xi^{(n)}(t)=X^{(n+1)}(t)-X^{(n)}(t),\\
&\Lambda_t^{(n)}=\sigma(t,X_t^{(n+1)},\mu_t^{(n)})-\sigma(t,X_t^{(n)},\mu_t^{(n-1)}),\\
&B_t^{(n)}=b(t,X_t^{(n+1)},\mu_t^{(n)})-b(t,X_t^{(n)},\mu_t^{(n-1)}).
\end{aligned}
$$

By $(H2)$ and Itô's formula, we have

$$
\mathrm{d}|\xi^{(n)}(t)|^2\le 2\langle\Lambda_t^{(n)}\mathrm{d}W(t),\xi^{(n)}(t)\rangle+K_1(t)\big\{\|\xi_t^{(n)}\|_\infty^2+\mathbb{W}_2(\mu_t^{(n)},\mu_t^{(n-1)})^2\big\}\mathrm{d}t.
$$

By the BDG inequality, we obtain

$$
\mathbb{E}\sup_{s\in[0,t]}|\xi^{(n)}(s)|^2
$$

$$\le c\mathbb{E}\bigg(\int_0^t \{|\xi^{(n)}(s)|^2\|\Lambda_s^{(n)}\|^2\}\mathrm{d}s\bigg)^{\frac{1}{2}} + K_1(t)\int_0^t \Big\{\mathbb{E}\|\xi_s^{(n)}\|_\infty^2 + \mathbb{W}_2(\mu_s^{(n)},\mu_s^{(n-1)})^2\Big\}\mathrm{d}s$$
$$\le \frac{1}{2}\mathbb{E}\sup_{s\in[0,t]}|\xi^{(n)}(s)|^2 + \frac{c^2}{2}\int_0^t \mathbb{E}\|\Lambda_s^{(n)}\|^2\mathrm{d}s + K_1(t)\int_0^t \Big\{\mathbb{E}\|\xi_s^{(n)}\|_\infty^2 + \mathbb{W}_2(\mu_s^{(n)},\mu_s^{(n-1)})^2\Big\}\mathrm{d}s$$

for some constant $c > 0$. This and $(H1)$ imply

$$\mathbb{E}\sup_{s\in[0,t]}|\xi^{(n)}(s)|^2 \le K(t)\int_0^t \Big\{\mathbb{E}\sup_{r\in[0,s]}|\xi^{(n)}(r)|^2 + \mathbb{W}_2(\mu_s^{(n)},\mu_s^{(n-1)})^2\Big\}\mathrm{d}s,\quad t\ge 0$$

for some increasing function $K:[0,\infty)\to(0,\infty)$. By Gronwall's lemma, we obtain

$$\mathbb{E}\sup_{s\in[0,t]}|\xi^{(n)}(s)|^2 \le tK(t)\mathrm{e}^{tK(t)}\sup_{s\in[0,t]}\mathbb{W}_2(\mu_s^{(n)},\mu_s^{(n-1)})^2 \le tK(t)\mathrm{e}^{tK(t)}\mathbb{E}\sup_{s\in[0,t]}|\xi^{(n-1)}(s)|^2,\quad t\ge 0.$$

Taking $t_0 > 0$ such that $t_0K(t_0)\mathrm{e}^{t_0K(t_0)} \le \mathrm{e}^{-1}$, we arrive at

$$\mathbb{E}\sup_{s\in[0,t_0]}|\xi^{(n)}(s)|^2 \le \mathrm{e}^{-1}\mathbb{E}\sup_{s\in[0,t_0]}|\xi^{(n-1)}(s)|^2,\quad n\ge 1.$$

Since

$$\mathbb{E}\sup_{s\in[0,t_0]}|\xi^{(0)}(s)|^2 \le 2\mathbb{E}\Big\{|X(0)|^2 + \sup_{s\in[0,t_0]}|X^{(1)}(s)|^2\Big\} \le 4\mathbb{E}\sup_{s\in[0,t_0]}|X^{(1)}(s)|^2,$$

this completes the proof. □

Proof of Theorem 2.1. Without loss of generality, we only consider the case from time $s=0$.

(1) Since the uniqueness follows from (2.2) which will be proved in the next step, in this step we only prove the existence and the estimate (2.1).

Since $L^2(\Omega, C([0,t_0];\mathbb{R}^d),\mathbb{P})$ is a Banach space, by Lemma 2.2, there exists an adapt continuous process $(X_t)_{t\in[0,t_0]}$ such that

$$\lim_{n\to\infty}\sup_{t\in[0,t_0]}\mathbb{W}_2(\mu_t^{(n)},\mu_t)^2 \le \lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,t_0]}|X^{(n)}(t)-X(t)|^2 = 0, \tag{2.7}$$

where μ_t is the distribution of X_t. Noting that due to (2.4)

$$X^{(n)}(t) = X(0) + \int_0^t b(s, X_s^{(n)}, \mu_s^{(n-1)})\mathrm{d}s + \int_0^t \sigma(s, X_s^{(n)}, \mu_s^{(n-1)})\mathrm{d}W(s),$$

it follows from (2.7), $(H1)$ and $(H3)$ and dominated convergence theorem that $\mathbb{P}$-a.s.

$$X(t) = X(0) + \int_0^t b(s, X_s, \mu_s)\mathrm{d}s + \int_0^t \sigma(s, X_s, \mu_s)\mathrm{d}W(s),\quad t\in[0,t_0].$$

Therefore, $(X_t)_{t\in[0,t_0]}$ is a solution to (1.5), and (2.7) implies $\mathbb{E}\sup_{s\in[0,t_0]}|X(s)|^2<\infty$. Since $t_0>0$ is independent of X_0, we conclude that (1.5) has a strong solution $(X_t)_{t\ge 0}$ with

$$\mathbb{E}\sup_{s\in[0,t]}|X(s)|^2 < \infty,\quad t\ge 0. \tag{2.8}$$

It remains to prove (2.1) for $\mathbb{E}\|X_0\|_\infty^{2p}<\infty$. By $(H2)$, $(H3)$ and Itô's formula we have

$$\mathrm{d}|X(t)|^2 \le 2\langle \sigma(t,X_t,\mathscr{L}_{X_t})\mathrm{d}W(t), X(t)\rangle + H(t)(1+\|X_t\|_\infty^2+\mathbb{E}\|X_t\|_\infty^2)\mathrm{d}t$$

for some increasing function $H:[0,\infty)\to(0,\infty)$. Let $\theta_n=\inf\{t\ge 0: |X(t)|\ge n\}$. Then by BDG inequality, for any $p\ge 1$ there exists a constant $C_p>0$ such that

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[-r_0,t\wedge\theta_n]}|X(s)|^{2p}\\
\le& C_p\mathbb{E}\|X_0\|_\infty^{2p} + C_p\mathbb{E}\bigg(\int_0^{t\wedge\theta_n} H(s)\big(1+\|X_s\|_\infty^2+\mathbb{E}\|X_s\|_\infty^2\big)\mathrm{d}s\bigg)^p\\
&+C_p\mathbb{E}\bigg(\int_0^{t\wedge\theta_n}|X(s)|^2\|\sigma(s,X_s,\mathscr{L}_{X_s})\|_{HS}^2\mathrm{d}s\bigg)^{\frac{p}{2}}\\
\le& C_p\mathbb{E}\|X_0\|_\infty^{2p} + C_p\mathbb{E}\bigg(\int_0^{t\wedge\theta_n} H(s)\big(1+\|X_s\|_\infty^2+\mathbb{E}\|X_s\|_\infty^2\big)\mathrm{d}s\bigg)^p\\
&+\frac{1}{2}\mathbb{E}\sup_{s\in[-r_0,t\wedge\theta_n]}|X(s)|^{2p} + \tilde{C}_p(t)\mathbb{E}\int_0^{t\wedge\theta_n}\|\sigma(s,X_s,\mathscr{L}_{X_s})\|_{HS}^{2p}\mathrm{d}s,\quad n\ge 1, t\ge 0.
\end{aligned}$$

Combining this with $(H3)$, we may find out an increasing function $C_p:[0,\infty)\to(0,\infty)$ such that

$$\begin{aligned}
\mathbb{E}\sup_{s\in[-r_0,t\wedge\theta_n]}|X(s)|^{2p} \le& 2C_p\mathbb{E}\|X_0\|_\infty^{2p}\\
&+C_p(t)\int_0^t\Big\{1+\mathbb{E}\sup_{u\in[-r_0,s\wedge\theta_n]}|X(u)|^{2p}+(\mathbb{E}\|X_s\|_\infty^2)^p\Big\}\mathrm{d}s,\quad n\ge 1, t\ge 0.
\end{aligned}$$

Since $\mathbb{E}\|X_0\|_\infty^{2p}<\infty$, Gronwall's lemma and (2.8) imply $\sup_{n\ge1}\mathbb{E}\sup_{s\in[-r_0,t\wedge\theta_n]}|X(s)|^{2p}<\infty$, so that by letting $n\to\infty$, we conclude that $h_t:=\mathbb{E}\sup_{s\in[-r_0,t]}|X(s)|^{2p}<\infty$ satisfies

$$h_t\le 2C_p\mathbb{E}\|X_0\|_\infty^{2p}+C_p(t)\int_0^t\big\{1+2h_s\big\}\mathrm{d}s,\quad t\ge 0.$$

By Gronwall's lemma, this implies estimate (2.1) for some increasing function H_p.

(2) By Itô's formula and $(H2)$, we have

$$\begin{aligned}
\mathrm{d}|X(t)-Y(t)|^2\le& 2\big\langle X(t)-Y(t),\{\sigma(t,X_t,\mathscr{L}_{X_t})-\sigma(t,Y_t,\mathscr{L}_{Y_t})\}\mathrm{d}W(t)\big\rangle\\
&+K_1(t)\big\{\|X_t-Y_t\|_\infty^2+\mathbb{W}_2(\mathscr{L}_{X_t},\mathscr{L}_{Y_t})^2\big\}\mathrm{d}t.
\end{aligned}$$

Noting that $\mathbb{W}_2(\mathscr{L}_{X_t},\mathscr{L}_{Y_t})^2\le\mathbb{E}\|X_t-Y_t\|_\infty^2$, combining with BDG inequality, it holds that

$$\mathbb{E}\sup_{s\in[-r_0,t]}|X(s)-Y(s)|^2\le 4\mathbb{E}\|X_0-Y_0\|_\infty^2+\int_0^t\tilde{K}(s)\mathbb{E}\sup_{r\in[-r_0,s]}|X(r)-Y(r)|^2\mathrm{d}s \tag{2.9}$$

for all $t>0$ and some increasing function $\tilde{K}:[0,\infty)\to(0,\infty)$. Since $\mathbb{E}\sup_{s\in[-r_0,t]}|X(s)-Y(s)|^2$ is locally bounded in t, Gronwall's lemma implies (2.2).

(3) Let $(X_t, W(t))$ w.r.t. $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ and $(\tilde{X}_t, \tilde{W}(t))$ w.r.t. $(\tilde{\Omega}, \{\tilde{\mathscr{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$ be two weak solutions such that $\mathscr{L}_{X_0}|_{\mathbb{P}} = \mathscr{L}_{\tilde{X}_0}|_{\tilde{\mathbb{P}}}$. Then X_t solves (1.5) while $\tilde{X}_t$ solves

$$\mathrm{d}\tilde{X}(t) = b(t, \tilde{X}_t, \mathscr{L}_{\tilde{X}_t}|_{\tilde{\mathbb{P}}})\mathrm{d}t + \sigma(t, \tilde{X}_t, \mathscr{L}_{\tilde{X}_t}|_{\tilde{\mathbb{P}}})\mathrm{d}\tilde{W}(t). \tag{2.10}$$

To prove that $\mathscr{L}_X|_{\mathbb{P}} = \mathscr{L}_{\tilde{X}}|_{\tilde{\mathbb{P}}}$, let $\mu_t = \mathscr{L}_{X_t}|_{\mathbb{P}}$ and

$$\bar{b}(t, \xi) = b(t, \xi, \mu_t), \quad \bar{\sigma}(t, \xi) = \sigma(t, \xi, \mu_t), \quad t \geq 0, \xi \in \mathscr{C}.$$

By $(H1)$-$(H3)$, the SDE

$$\mathrm{d}\bar{X}(t) = \bar{b}(t, \bar{X}_t)\mathrm{d}t + \bar{\sigma}(t, \bar{X}_t)\mathrm{d}\tilde{W}(t), \quad \bar{X}_0 = \tilde{X}_0 \tag{2.11}$$

has a unique solution for any initial points. According to Yamada-Watanabe, it also has weak uniqueness. Noting that

$$\mathrm{d}X(t) = \bar{b}(t, X_t)\mathrm{d}t + \bar{\sigma}(t, X_t)\mathrm{d}W(t), \quad \mathscr{L}_{X_0}|_{\mathbb{P}} = \mathscr{L}_{\tilde{X}_0}|_{\tilde{\mathbb{P}}},$$

the weak uniqueness of (2.11) implies

$$\mathscr{L}_{\bar{X}}|_{\tilde{\mathbb{P}}} = \mathscr{L}_X|_{\mathbb{P}}. \tag{2.12}$$

So, (2.11) reduces to

$$\mathrm{d}\bar{X}(t) = b(t, \bar{X}_t, \mathscr{L}_{\bar{X}_t}|_{\tilde{\mathbb{P}}})\mathrm{d}t + \sigma(t, \bar{X}_t, \mathscr{L}_{\bar{X}_t}|_{\tilde{\mathbb{P}}})\mathrm{d}\tilde{W}(t), \quad \bar{X}_0 = \tilde{X}_0.$$

Since by (1) (2.10) has a unique solution, we obtain $\bar{X} = \tilde{X}$. Therefore, the weak uniqueness follows from (2.12).

Finally, since $\mathscr{C}$ is a Polish space, then for any $\mu_0, \nu_0 \in \mathscr{P}_2$, we can take $\mathscr{F}_0$-measurable random variables X_0, Y_0 such that $\mathscr{L}_{X_0} = \mu_0, \mathscr{L}_{Y_0} = \nu_0$ and $\mathbb{W}_2(\mu_0, \nu_0)^2 = \mathbb{E}\|X_0 - Y_0\|_\infty^2$. Since $\mathbb{W}_2(P_t^*\mu_0, P_t^*\nu_0)^2 \leq \mathbb{E}\|X_t - Y_t\|_\infty^2$, (2.2) implies (2.3).

□

3 Harnack inequality and applications

Using coupling by change of measure, Harnack inequalities are obtained for various SDEs and SPDEs, e.g. [1, 2, 3, 5, 6, 8, 9, 11, 16]. In this section, we investigate the dimension-free Harnack inequality with power and log-Harnack inequality introduced in [5, 8] for the DDSDDE (1.5), see [7] for the Harnack inequality for functional S(P)DEs. We establish Harnack inequalities for $P_t f$ (see for instance [7, §1.1]), we need to assume that the noise part is distribution-free and without delay. More over, in order to use the conclusions in [7, Theorem 4.3.1, 4.3.2], we only consider the following special version

$$\mathrm{d}X(t) = b(t, X_t, \mu_t)\mathrm{d}t + \sigma(t, X(t))\mathrm{d}W(t), \quad \mathscr{L}_{X_0} = \mu_0. \tag{3.1}$$

See the proof of Theorem 3.1 in the following for the reason why we replace [7, 4.17] with (3.1).

Define

$$(P_tf)(\mu_0)=\int_{\mathscr{C}} f\mathrm{d}(P_t^*\mu_0)=\mathbb{E}f(X_t^{\mu_0}),\quad f\in\mathscr{B}_b(\mathscr{C}),t\geq 0,\mu_0\in\mathscr{P}_2,$$

where $X_t^{\mu_0}$ solves (3.1) with initial distribution μ_0.

In this section, we do not repeat the coupling by change of measure, instead, we use the Girsanov's theorem and the Harnack inequality for the classical SFDEs, see [7, Theorem 4.3.1, 4.3.2]. We need the following assumption.

(A) $\sigma(t,x)$ is invertible and locally Lipschitzian in x which is locally uniformly in $t\geq 0$, $\|\sigma(t,\cdot)\|_\infty$ is locally bounded in t, and there exist increasing functions $\kappa_0,\kappa_1,\kappa_2,\lambda:[0,\infty)\to(0,\infty)$ such that for any $t\in[0,T],x,y\in\mathbb{R}^d$ and $\mu,\nu\in\mathscr{P}_2$, we have

$$\|\sigma(t,\cdot)^{-1}\|_\infty\leq\lambda(t),\quad |b(t,0,\mu)|^2+\|\sigma(t,x)\|^2\leq\kappa_0(t)(1+|x|^2+\mu(\|\cdot\|_\infty^2)),\tag{3.2}$$

$$\|\sigma(t,x)-\sigma(t,y)\|_{HS}^2\leq\kappa_1(t)|x-y|^2,\tag{3.3}$$

$$\|\sigma(t,\cdot)\|_\infty\leq\kappa_3(t).\tag{3.4}$$

$$|b(t,\xi,\mu)-b(t,\eta,\nu)|\leq\kappa_4(t)(\|\xi-\eta\|_\infty+\mathbb{W}_2(\mu,\nu)).\tag{3.5}$$

3.1 Main results

Similar to [7, §4.3], we have the following results.

Theorem 3.1. *Assume* **(A)***. Then for any $\mu_0,\nu_0\in\mathscr{P}_2$ and $\mathscr{F}_0$-measurable random variables X_0,Y_0 with $\mathscr{L}_{X_0}=\mu_0,\mathscr{L}_{Y_0}=\nu_0$, $T>r_0$ and positive function $f\in\mathscr{B}_b(\mathscr{C})$,*

(1) the log*-Harnack inequality holds, i.e.*

$$(P_T\log f)(\nu_0)\leq\log(P_Tf)(\mu_0)+\mathbb{E}H_1(T,X_0,Y_0)+H_2(T)\mathbb{W}_2(\mu_0,\nu_0)^2.\tag{3.6}$$

In particular,

$$P_T\log f(\eta)\leq\log P_Tf(\xi)+H(T,\xi,\eta)+H_2(T)\|\xi-\eta\|_\infty^2,\ \ \xi,\eta\in\mathscr{C}\tag{3.7}$$

where

$$H_1(T,\xi,\eta)=C\left(\frac{|\xi(0)-\eta(0)|^2}{T-r_0}+\|\xi-\eta\|_\infty^2\right)$$

and

$$H_2(T)=\int_0^T\lambda(t)^2\kappa_4(t)^2\mathrm{e}^{2\int_0^t(3\kappa_4(s)+\kappa_1(s))\mathrm{d}s}\mathrm{d}t$$

for some constant $C>0$.

(2) There exists $p(T)>0$ such that for any $p>p(T)$, the Harnack inequality with power

$$(P_Tf)(\nu_0)\leq(P_{s,t}f^p)^{\frac{1}{p}}(\mu_0)\mathbb{E}\exp\Psi_p(T;X_0,Y_0)\times\exp\{\tilde{\Psi}(p,T)\mathbb{W}_2(\mu_0,\nu_0)^2\}. \tag{3.8}$$

holds. In particular,

$$P_Tf(\eta)\leq(P_Tf^p(\xi))^{\frac{1}{p}}\exp\Psi_p(T;\xi,\eta)\times\exp\{\tilde{\Psi}(p,T)\|\xi-\eta\|_\infty^2\},\quad \xi,\eta\in\mathscr{C}, \tag{3.9}$$

where

$$\begin{aligned}\Psi_p(T;\xi,\eta)&=C(p)\left\{1+\frac{|\xi(0)-\eta(0)|^2}{T-r_0}+\|\xi-\eta\|_\infty^2\right\},\\ \tilde{\Psi}(p,T)&=\frac{1}{p-\sqrt{p}}\int_0^T\lambda(t)^2\kappa_4(t)^2\mathrm{e}^{2\int_0^t(3\kappa_4(s)+\kappa_1(s))\mathrm{d}s}\mathrm{d}t\end{aligned}$$

for a decreasing function $C:(p(T),\infty)\to(0,\infty)$.

Below we present some direct consequence of the above Harnack inequalities.

Corollary 3.2. *Assume* **(A)** *and let $T>r_0$. For any $\mu_0,\nu_0\in\mathscr{P}_2$, $P_T^*\mu_0$ and $P_T^*\nu_0$ are equivalent and the Radon-Nykodim derivative satisfies the entropy estimate*

$$\int_{\mathscr{C}}\left\{\log\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}P_T^*\mu_0}\right\}\mathrm{d}P_T^*\nu_0\leq\mathbb{E}H_1(T,X_0,Y_0)+H_2(T)\mathbb{W}_2(\mu_0,\nu_0)^2 \tag{3.10}$$

and for any $p\geq p(T)$,

$$\int_{\mathscr{C}}\left\{\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}P_T^*\mu_0}\right\}^{\frac{1}{p}}\mathrm{d}(P_T^*\nu_0)\leq\mathbb{E}\exp\Psi_p(T;X_0,Y_0)\times\exp\{\tilde{\Psi}(p,T)\mathbb{W}_2(\mu_0,\nu_0)^2\}, \tag{3.11}$$

where X_0,Y_0 are $\mathscr{F}_0$-measurable random variables with $\mathscr{L}_{X_0}=\mu_0,\mathscr{L}_{Y_0}=\nu_0$.

Proof. See the proof of [14, Corollary 4.3] for details. □

3.2 Proof of Theorem 3.1

We write (3.1) as

$$\mathrm{d}X(t)=\bar{b}(t,X_t)\mathrm{d}t+\sigma(t,X(t))\mathrm{d}\bar{W}(t). \tag{3.12}$$

where

$$\begin{aligned}&\bar{b}(t,\xi)=b(t,\xi,\nu_t),\quad \mathrm{d}\bar{W}(t)=\mathrm{d}W(t)+\bar{\gamma}(t)\mathrm{d}t,\\ &\bar{\gamma}(t)=\sigma^{-1}(t,X(t))[b(t,X_t,\mu_t)-b(t,X_t,\nu_t)].\end{aligned}$$

By (3.2) and (3.5), we have

$$|\bar{\gamma}(t)|\leq\lambda(t)\kappa_4(t)\mathbb{W}_2(\mu_t,\nu_t),\quad t\in[0,T]. \tag{3.13}$$

Let

$$\bar{R}_t = \exp\left\{-\int_0^t \langle\bar{\gamma}(s), \mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^t |\bar{\gamma}(s)|^2 \mathrm{d}s\right\}, \quad t \in [0,T].$$

Define $\bar{\mathbb{P}}_t = \bar{R}_t\mathbb{P}$. Then $\{\bar{W}(t)\}_{t\in[0,T]}$ is a d-dimensional Brownian motion under $\bar{\mathbb{P}}_T$. According to [7], we can construct a coupling process $\{Y(t)\}_{t\in[-r_0,T]}$ such that

$$\mathrm{d}Y(t) = \bar{b}(t, Y_t)\mathrm{d}t + \sigma(t, Y(t))\mathrm{d}\tilde{W}(t), \quad \mathscr{L}_{Y_0} = \nu_0 \tag{3.14}$$

where

$$\mathrm{d}\tilde{W}(t) = \mathrm{d}\bar{W}(t) + \tilde{\gamma}(t)\mathrm{d}t = \mathrm{d}W(t) + [\bar{\gamma}(t) + \tilde{\gamma}(t)]\mathrm{d}t,$$

for some process $\tilde{\gamma}(t)$. Letting

$$\tilde{R}_t = \exp\left\{-\int_0^t \langle\tilde{\gamma}(s), \mathrm{d}\bar{W}(s)\rangle - \frac{1}{2}\int_0^t |\tilde{\gamma}(s)|^2 \mathrm{d}s\right\}, \quad t \in [0,T].$$

and $\tilde{\mathbb{Q}}_t = \tilde{R}_t\bar{\mathbb{P}} = \tilde{R}_t\bar{R}_t\mathbb{P} =: R_t\mathbb{P}$, then $\{\tilde{W}(t)\}_{t\in[0,T]}$ is a d-dimensional standard Brownian motion under $\tilde{\mathbb{Q}}_T$, and $X_T = Y_T, \tilde{\mathbb{Q}}_T$-a.s.. Moreover, it is easy to see that

$$R_t = \exp\left\{-\int_0^t \langle\bar{\gamma}(s) + \tilde{\gamma}(s), \mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^t |\bar{\gamma}(s) + \tilde{\gamma}(s)|^2 \mathrm{d}s\right\}, \quad t \in [0,T]. \tag{3.15}$$

Let

$$\mathrm{d}\tilde{X}(t) = b(t, \tilde{X}_t, \mathscr{L}_{\tilde{X}_t}|\tilde{\mathbb{Q}}_T)\mathrm{d}t + \sigma(t, \tilde{X}(t))\mathrm{d}\tilde{W}(t), \quad \mathscr{L}_{\tilde{X}_0} = \nu_0. \tag{3.16}$$

according to the weak uniqueness, we have $\mathscr{L}_{\tilde{X}_t}|\tilde{\mathbb{Q}}_T = P_t^*\nu_0 = \nu_t$, then by the strong uniqueness for the classical SDE, we have $Y_t = \tilde{X}_t$ which implies $\mathscr{L}_{Y_t}|\tilde{\mathbb{Q}}_T = \nu_t$.

Then for any $f \in \mathscr{B}_b^+(\mathscr{C})$,

$$(P_Tf)(\nu_0) = \mathbb{E}_{\tilde{\mathbb{Q}}}[f(Y_T)] = \mathbb{E}[R_Tf(X_T)], \quad f \in \mathscr{B}_b(\mathscr{C}).$$

So, by Young's inequality and Hölder's inequality respectively, we obtain

$$(P_T \log f)(\nu_0) \le \mathbb{E}[R_T \log R_T] + \log \mathbb{E}[f(X_T)] = \mathbb{E}[R_T \log R_T] + \log(P_Tf)(\mu_0) \tag{3.17}$$

and

$$(P_Tf(\nu_0))^p \le (\mathbb{E}R_T^{\frac{p}{p-1}})^{p-1}(\mathbb{E}f^p(X_T)) = (\mathbb{E}R_T^{\frac{p}{p-1}})^{p-1}P_Tf^p(\mu_0) \tag{3.18}$$

for any $p > 1$.

Next, according to [7, Theorem 4.3.1], (3.13) and (3.15), we have

$$
\begin{aligned}
\mathbb{E}[R_T \log R_T] &\le \frac{1}{2}\mathbb{E}_{\tilde{\mathbb{Q}}_T}\int_0^T |\bar\gamma(s)+\tilde\gamma(s)|^2 \mathrm{d}s \\
&\le \mathbb{E}_{\tilde{\mathbb{Q}}_T}\int_0^T |\tilde\gamma(s)|^2 \mathrm{d}s + \int_0^T |\bar\gamma(s)|^2 \mathrm{d}s \\
&\le \mathbb{E}_{\tilde{\mathbb{Q}}_T}\int_0^T |\tilde\gamma(s)|^2 \mathrm{d}s + \int_0^T \lambda(t)^2\kappa_4(t)^2\mathbb{W}_2(\mu_t,\nu_t)^2 \mathrm{d}t \\
&\le \mathbb{E}H(T,X_0,Y_0) + \int_0^T \lambda(t)^2\kappa_4(t)^2\mathbb{W}_2(\mu_0,\nu_0)^2 \mathrm{e}^{2\int_0^t(3\kappa_4(s)+\kappa_1(s))\mathrm{d}s}\mathrm{d}t
\end{aligned} \tag{3.19}
$$

for any $\mathscr{F}_0$-measurable random variables X_0, Y_0 with $\mathscr{L}_{X_0}=\mu_0, \mathscr{L}_{Y_0}=\nu_0$. H is in Theorem 3.1.

Moreover, by the definition of R_T and $\tilde{\mathbb{Q}}_T$, [7, Theorem 4.3.2], (3.13) and (3.15), there exists a constant $p(T)>0$ such that for any $p>p(T)$, it holds that

$$
\begin{aligned}
(\mathbb{E}R_T^{\frac{p}{p-1}})^{\frac{p-1}{p}} &\le \left(\mathbb{E}_{\tilde{\mathbb{Q}}_T}\exp\left[\frac{(\sqrt{p}+1)(p+\sqrt{p})}{2(p-1)^2\sqrt{p}}\int_0^T |\bar\gamma(s)+\tilde\gamma(s)|^2\mathrm{d}s\right]\right)^{\frac{\sqrt{p}-1}{\sqrt{p}}} \\
&\le \left(\mathbb{E}_{\tilde{\mathbb{Q}}_T}\exp\left[\frac{(\sqrt{p}+1)(p+\sqrt{p})}{(p-1)^2\sqrt{p}}\int_0^T |\tilde\gamma(s)|^2\mathrm{d}s\right]\right)^{\frac{\sqrt{p}-1}{\sqrt{p}}} \\
&\quad\times \exp\left[\frac{1}{p-\sqrt{p}}\int_0^T \lambda(t)^2\kappa_4(t)^2\mathbb{W}_2(\mu_t,\nu_t)^2\mathrm{d}t\right] \\
&\le \mathbb{E}\exp\Psi_p(T;X_0,Y_0)\times\exp\{\tilde\Psi(p,T)\mathbb{W}_2(\mu_0,\nu_0)^2\},
\end{aligned} \tag{3.20}
$$

where $\Psi_p(T;\xi,\eta)$ and $\tilde\Psi(p,T)$ are in Theorem 3.1. Substitute (3.19) and (3.20) into (3.17) and (3.18) respectively, we obtain Theorem 3.1.

4 Shift Harnack Inequality and Integration by Parts Formula

So far, with backward coupling by change of measure or Mallivian calculus, there are many results on the shift Harnack inequalities and integration by parts formula for the S(P)DEs with additive noise, see [4, 10, 12, 13, 7, 15]. In this section we establish the shift Harnack inequality and integration by parts formula introduced in [10]. Since the study for the multiplicative noise case is very complicated, here we only consider the additive noise for which the DDSFDE (1.5) reduces to

$$
\mathrm{d}X(t) = b(t,X_t,\mathscr{L}_{X_t})\mathrm{d}t + \sigma(t)\mathrm{d}W(t). \tag{4.1}
$$

The following comes from [7, §4.2.2] for references. It turns out that we are able to make derivatives or shift only along directions in the Cameron-Martin space

$$\mathbb{H}^1 := \left\{ h \in \mathscr{C}, \|h\|_{\mathbb{H}^1}^2 := \int_{-r_0}^0 |h'(t)|^2 \mathrm{d}t < \infty \right\}.$$

Theorem 4.1. *Let $\sigma : [0,\infty) \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : [0,\infty) \times \mathscr{C} \times \mathscr{P}_2 \to \mathbb{R}^d$ are measurable such that σ is invertible with $\|\sigma(t)\| + \|\sigma(t)^{-1}\|$ locally bounded in $t \geq 0$, and $b(t,\cdot,\mu_t)$ is differentiable with*

$$\kappa(T) := \sup_{t\in[0,T], \|\xi\|_\infty \leq 1} \|\nabla_\xi b(t,\cdot,\mu_t)\|_\infty < \infty, \quad T \geq 0, \mu_\cdot \in C([0,T];\mathscr{P}_2).$$

Let $\Sigma(T) := \sup_{t\in[0,T]} \|\sigma(t)^{-1}\|^2$, $T \geq 0$. For every $\alpha^{s,t} \in \mathscr{B}_b([s, t-r_0])$ with $\int_s^{t-r_0} \alpha(u)\mathrm{d}u = 1$, let

$$\begin{aligned}
\Phi^{s,t}(u) &= 1_{[s,t-r_0]}(u)\alpha^{s,t}(u)\eta(-r_0) + 1_{(t-r_0,t]}(u)\eta'(u-t), \\
\Theta^{s,t}(u) &= \int_s^{u\vee s} \Phi^{s,t}(u)\mathrm{d}u, \quad u \in [s-r_0, t].
\end{aligned}$$

(1) *For any $p > 1, t > s + r_0 \geq r_0, \mu_0 \in \mathscr{P}_2, \eta \in \mathbb{H}^1$ and $f \in \mathscr{B}_b^+(\mathscr{C})$,*

$$\begin{aligned}
(P_{s,t}f)^p(\mu_0) \leq &(P_{s,t}f^p(\eta + \cdot))(\mu_0) \\
&\times \exp\left[\frac{p\,\Sigma(t)\,(1+(t-s)^2\kappa^2(t))\left(\frac{|\eta(-r_0)|^2}{t-r_0} + \|\eta\|_{\mathbb{H}^1}^2\right)}{(p-1)^2}\right].
\end{aligned}$$

Moreover,

$$\begin{aligned}
(P_{s,t}\log f)(\mu_0) &\leq \log(P_{s,t}f(\eta+\cdot))(\mu_0) \\
&\quad + \Sigma(t)\left(1+(t-s)^2\kappa^2(t)\right)\left(\frac{|\eta(-r_0)|^2}{t-r_0} + \|\eta\|_{\mathbb{H}^1}^2\right).
\end{aligned}$$

(2) *For any $t > s + r_0 \geq r_0, f \in C_b^1(\mathscr{C})$ and $\mathscr{F}_s$-measurable random variable X_s^s with $\mu_0 := \mathscr{L}_{X_s^s} \in \mathscr{P}_2$, $\eta \in \mathbb{H}^1$, it holds that*

$$\mathbb{E}(\nabla_\eta f)(X_t^s) = \mathbb{E}\left[f(X_t^s)\int_s^t \left\langle \sigma(r)^{-1}(\Phi^{s,t}(r) - \nabla_{\Theta_r^{s,t}} b(r,\cdot,P_{s,r}^*\mu_0)(X_r^s)),\ \mathrm{d}W(r)\right\rangle\right].$$

Proof. (1) Without loss of generality, we only prove for $s = 0$ and $t = T$ for some fixed time $T > r_0$. For simplicity, we denote $\alpha = \alpha^{0,T}$, $\Phi = \Phi^{0,T}$ and $\Theta = \Theta^{0,T}$. Denote $\mu_t = P_t^*\mu_0 = \mathscr{L}_{X_t}, t \geq 0$. Then (4.1) becomes

$$\mathrm{d}X(t) = b(t, X_t, \mu_t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t), \quad \mathscr{L}_{X_0} = \mu_0. \tag{4.2}$$

$$\Phi(t) = 1_{[0,T-r_0]}(t)\alpha(t)\eta(-r_0) + 1_{(T-r_0,T]}(t)\eta^{'}(t-T), \quad \Theta(t) = \int_0^{t\vee 0} \Phi(s)\mathrm{d}s, \quad t \in [-r_0, T].$$

Let $Y(t) = X(t) + \Theta(t),\ t \in [-r_0, T]$. Then

$$\mathrm{d}Y(t) = b(t, Y_t, \mu_t)\mathrm{d}t + \sigma(t)\mathrm{d}\tilde{W}(t), \quad \mathscr{L}_{Y_0} = \mu_0, t \in [0, T], \tag{4.3}$$

where

$$\begin{aligned} &\tilde{W}(t) := W(t) + \int_0^t \beta_s \mathrm{d}s, \\ &\beta_t := \sigma(t)^{-1}\Big\{\Phi(t) + b(t, X_t, \mu_t) - b(t, X_t + \Theta_t, \mu_t)\Big\}. \end{aligned}$$

Let $R_T = \exp[-\int_0^T \langle \beta_t, \mathrm{d}W(t)\rangle - \frac{1}{2}\int_0^T |\beta_t|^2 \mathrm{d}t]$, then by Girsanov theorem, $\{\tilde{W}(t)\}_{t\in[0,T]}$ is a d-dimensional Brownian motion under the probability $\mathrm{d}\mathbb{Q}_T := R_T \mathrm{d}\mathbb{P}$.

Consider the DDSFDE

$$\mathrm{d}\tilde{X}(t) = b(t, \tilde{X}_t, \mathscr{L}_{\tilde{X}_t}|\mathbb{Q}_T)\mathrm{d}t + \sigma(t)\mathrm{d}\tilde{W}(t), \quad \tilde{X}_0 = Y_0.$$

Since $R_0 = 1$, by the weak uniqueness we have $\mathscr{L}_{\tilde{X}_t}|\mathbb{Q}_T = P_t^*\mu_0 = \mu_t$ for $t \in [0, T]$. Combining this with (4.3) and the strong uniqueness, we conclude that $\tilde{X}_t = Y_t$ for $t \in [0, T]$. Thus

$$(P_T f)(\mu_0) = \mathbb{E}[R_T f(Y_T)] = \mathbb{E}[R_T f(X_T + \eta)] \le (P_T f^p(\eta + \cdot))^{\frac{1}{p}}(\mu_0)\big(\mathbb{E}R_T^{\frac{p}{p-1}}\big)^{\frac{p-1}{p}}.$$

Take $\alpha(t) = \frac{1}{T-r_0}$, we have

$$\begin{aligned} &\int_0^T |\Phi(t)|^2 \mathrm{d}t = \frac{|\eta(-r_0)|^2}{T - r_0} + \|\eta\|_{\mathbb{H}^1}^2, \\ &\|\nabla_{\Theta_t} b(t, \cdot, \mu_t)\|_\infty^2 \le \kappa^2 T \int_0^T |\Phi(t)|^2 \mathrm{d}t = T\left(\frac{|\eta(-r_0)|^2}{T - r_0} + \|\eta\|_{\mathbb{H}^1}^2\right), \quad t \in [0, T]. \end{aligned}$$

Then

$$\begin{aligned} &\left\{\int_0^T \|\sigma(t)^{-1}\|^2 \left(|\Phi(t)|^2 + \|\nabla_{\Theta_t} b(t, \cdot, P_t^*\mu_0)\|_\infty^2\right) \mathrm{d}t\right\} \\ &\le \sup_{t\in[0,T]} \|\sigma(t)^{-1}\|^2 \left(1 + \kappa^2 T^2\right)\left(\frac{|\eta(-r_0)|^2}{T - r_0} + \|\eta\|_{\mathbb{H}^1}^2\right). \end{aligned} \tag{4.4}$$

Let $N_T = \int_0^T \langle \beta_t, \mathrm{d}W(t)\rangle$. According to the definition of R_T and (4.4), we have

$$\begin{aligned} \mathbb{E}R_T^{\frac{p}{p-1}} &= \mathbb{E}\exp\left[-\frac{p}{p-1}N_T - \frac{p^2}{2(p-1)^2}\langle N\rangle_T + \frac{p}{2(p-1)^2}\langle N\rangle_T\right] \\ &\le \mathbb{E}\exp\left[-\frac{p}{p-1}N_T - \frac{p^2}{2(p-1)^2}\langle N\rangle_T\right] \end{aligned}$$

$$\times \exp\left[\frac{2p\int_0^T \|\sigma(t)^{-1}\|^2\left(|\Phi(t)|^2 + \|\nabla_{\Theta_t} b(t,\cdot,P_t^*\mu_0)\|_\infty^2\right)\mathrm{d}t}{2(p-1)^2}\right]$$

$$\le \exp\left[\frac{p\sup_{t\in[0,T]}\|\sigma(t)^{-1}\|^2\left(1+\kappa^2T^2\right)\left(\frac{|\eta(-r_0)|^2}{T-r_0} + \|\eta\|_{\mathbb{H}^1}^2\right)}{(p-1)^2}\right].$$

Similarly,

$$\begin{aligned}
\mathbb{E}R_T\log R_T &= \frac{1}{2}\mathbb{E}_{\mathbb{Q}_T}\int_0^T |\beta_t|^2\mathrm{d}t \\
&\le \int_0^T \|\sigma(t)^{-1}\|^2\left(|\Phi(t)|^2 + \|\nabla_{\Theta_t} b(t,\cdot,P_t^*\mu_0)\|_\infty^2\right)\mathrm{d}t \\
&\le \sup_{t\in[0,T]}\|\sigma(t)^{-1}\|^2\left(1+\kappa^2T^2\right)\left(\frac{|\eta(-r_0)|^2}{T-r_0} + \|\eta\|_{\mathbb{H}^1}^2\right).
\end{aligned}$$

This proves (1).

To prove (2), we let $X^\varepsilon(t) = X(t) + \varepsilon\Theta(t)$ for $\varepsilon\in(0,1)$ and $t\in[-r_0,T]$. Using $\varepsilon\eta$ replace η, the above argument implies

$$(P_Tf)(\mu_0) = \mathbb{E}[R_T^\varepsilon f(X_T + \varepsilon\eta)],\quad \varepsilon\in(0,1),$$

where

$$\begin{aligned}
R_T^\varepsilon &:= \exp\left[-\int_0^T \langle \xi_t^\varepsilon, \mathrm{d}W(t)\rangle - \frac{1}{2}\int_0^T |\xi_s^\varepsilon|^2\mathrm{d}s\right], \\
\xi_t^\varepsilon &:= \sigma(t)^{-1}\Big\{\varepsilon\Phi(t) + b(t,X_t,\mu_t) - b(t,X_t+\varepsilon\Theta_t,\mu_t)\Big\}.
\end{aligned}$$

Since $\{R_T^\varepsilon\}_{\varepsilon\in[0,1]}$ is uniformly integrable, we have

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}[R_T^\varepsilon f(X_T+\varepsilon\eta) - f(X_T)] \\
&= \mathbb{E}[(\nabla_\eta f)(X_T)] - \mathbb{E}\bigg[f(X_T)\int_0^T \Big\langle \sigma(r)^{-1}(\Phi(r) - \nabla_{\Theta_r} b(r,\cdot,P_r^*\mu_0)(X_r)),\ \mathrm{d}W(r)\Big\rangle\bigg].
\end{aligned}$$

Then the proof is finished.

Remark 4.1 In fact, if we let $\bar(b)(t,\xi) = b(t,\xi,\mu_t)$, then (4.1) is a classical SFDE. Then, by using the results in [7, Theorem 4.2.1, 4.2.2] directly, we can also obtain Theorem 4.1. $\square$

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